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THE EXISTENCE OF EQUILIBRIA IN CERTAIN GAMES, SEPARATION FOR FAMILIES OF CONVEX FUNCTIONS AND A THEOREM OF BORSUK–ULAM TYPE

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ABSTRACT

The existence of a Nash equilibrium in the undiscounted repeated two-person game of incomplete information on one side is established. The proof depends on a new topological result resembling in some respect the Borsuk–Ulam theorem.

Introduction

The motivation for considering results of this note comes from game theory, and specifically from the problem, posed in 1968 by R. Aumann, M. Maschler and R. Stearns [Au-Ma-St], whether any undiscounted infinitely repeated two-person game of incomplete information on one side has a Nash equilibrium. (See also [Fo, §3.3]. A short description of these games and of the necessary notions is given in §3.) In this context these authors defined and studied a certain class of potential

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equilibria, which they called "joint plan equilibria". In 1983 S. Sorin considered a modification of a special case of the joint plan equilibrium and proved that his "independent and 2-safe" joint plan equilibrium exists whenever the game has only two states of nature. In the present note we settle in the positive the problem stated above by proving the existence of such an equilibrium for games with an arbitrary number of states of nature.

Our proof is a generalization of Sorin's 1983 approach. It depends on a topological result; this is not a surprise, as several proofs in game theory depend on various fixed-point and related theorems. Here we need however a new result, which in its special case can be stated as follows: If x_0 is a point of a compact set $C \subset \mathbb{R}^n$ and $f\colon C \to Y$ is a mapping into a space of dimension $n-1$, then in the boundary of C there exists a set C_0 mapped by f into a singleton and containing x_0 in its convex hull. We say this is a statement of Borsuk–Ulam type, as in the special case where C is a disk and Y an Euclidean space the Borsuk–Ulam theorem shows that one may take for C_0 a set consisting of 2 points. In fact we show that a result of Olędzki easily implies that C_0 may be be taken to consist of 2 points whenever Y is a manifold; however for our applications it is essential to avoid assuming the latter and also to allow f to be a multifunction. (By Carathéodory's theorem one can always replace C_0 by its subset consisting of $\leq n+1$ points, but in general we do not know when this bound may be lowered.)

The paper is organized as follows. We treat the topological results in §1 and undiscounted repeated games in §3. In §2 we solve a certain system of inequalities needed in §3 to prove the existence of equilibria. This is closely related to proving in §2 a more general Theorem 2 giving conditions on a family $\{b_v\colon Q \to \mathbb{R}\}_{v \in \mathbb{R}^n}$ of convex functions and on a convex set $P \subset \mathbb{R}^n$ which suffice for the following to be true: given a $p_0 \in P$ and a function $a\colon Q \to \mathbb{R}$ such that $a \leq b_v$ $\forall v \in \mathbb{R}^n$, there exists a set $P_0 \subset \partial P$ containing p_0 in its convex hull and vectors v_p normal to ∂P at p ($p \in P_0$) such that an affine functional on $\mathbb{R}^n$ separates a from any of the functions b_{p+v_p}, $p \in P_0$. Theorem 2 is established by using results of §1 and is a geometric counterpart to the game-theoretic statements of this paper.

Notation: If K is a finite set then $|K|$ denotes its cardinality, $\mathbb{R}^K$ and $\mathbb{R}^{|K|}$ the euclidean $|K|$-space, equipped with the standard norm, e_k the coordinate unit vectors of $\mathbb{R}^K$, $\Delta(K)$ the simplex $\operatorname{conv}\{e_k\colon k \in K\}$ and e the vector $\sum_{k \in K} e_k$. The k-th coordinate of a vector $y \in \mathbb{R}^K$ is denoted $y(k)$ or y^k. If C is subset of an euclidean space then ∂C and $\operatorname{int} C$ denote, respectively, the boundary and the interior of C relative its affine span. All spaces are assumed to be metrizable.

1. A result of Borsuk–Ulam type

By a multifunction $\Phi\colon X \to Y$ we understand here a set $\Phi \subset X \times Y$ such that for every $x \in X$ the set $\Phi(x) := \{y \in Y\colon (x,y) \in \Phi\}$ is nonempty and compact. We write $\Phi^{-1}(y) := \{x \in X\colon (x,y) \in \Phi\}$ and $\Phi(A) := \bigcup\{\Phi(a)\colon a \in A\}$ for $y \in Y$ and $A \subset X$. Φ is said to be upper-semicontinuous if $\Phi \cap (A \times Y)$ is compact for every compact subset A of X. (This holds whenever Φ is closed in $X \times Y$ and Y is compact.)

Below, we need additional assumptions on the multifunctions we consider. The case we have in mind is when they take values in contractible compacta, but to treat this we fix a homology or cohomology functor h defined on the category of all compacta and request merely that the multifunctions $\Phi\colon X \to Y$ under consideration are n-acyclic, i.e. that $h_k(\Phi(x)) = 0$ for all $k < n$ whenever $x \in X$. We need h to satisfy standard axioms on the category of polyhedra and to be continuous for inverse limits ([E-S] and [Sp]); also, we request that Vietoris' theorem ([Be] and [Sp]) be true for h. Thus h may be the reduced Čech homology with coefficients in a compact field or Čech cohomology; we write our proofs for homology and they may be dualized easily.

Setup: In this section we denote by C a compact subset of $\mathbb{R}^n$ of dimension n and we fix a point $x_0 \in C$ and upper-semicontinuous multifunctions $F\colon C \to Y$ and $G\colon F(\partial C) \to \mathbb{R}^n$.

THEOREM 1: *Suppose F and G are n-acyclic and* $\dim(F(C) \setminus F(\partial C)) \leq n-1$. *If $G(y) \supset F^{-1}(y) \cap \partial C$ for every $y \in F(\partial C)$ then there exists a point $y_0 \in F(\partial C)$ such that $x_0 \in G(y_0)$.*

Remark 1: We have $\dim(F(C) \setminus F(\partial C)) \leq n-1$ whenever $\dim F(\operatorname{int} C) \leq n-1$. ∎

For the proof of the theorem we need two lemmas:

LEMMA 1: *If Y is compact and Y' is a closed subset of Y such that* $\dim(Y \setminus Y')$ $\leq n-1$, *then $h_{n-1}(Y' \hookrightarrow Y)$ is a monomorphism.*

Proof: Let $(\mathbf{Y}, \mathbf{Y}') = ((Y_k, Y'_k), \pi^l_k, N)$ be an inverse sequence of polyhedral pairs such that (Y, Y') is the inverse limit of $(\mathbf{Y}, \mathbf{Y}')$ and $\dim(Y_k \setminus Y'_k) \leq n-1$

for every k. By the homology exact sequence we have $\ker h_{n-1}(Y'_k \hookrightarrow Y_k) = h_n(Y_k, Y'_k) = 0$. Since inverse limits preserve monomorphisms the lemma follows from the continuity of the functor h. ∎

Lemma 2: *There exists an $a \in h_{n-1}(\partial C)$ such that for any compact subset K of $\mathbb{R}^n$ containing ∂C the properties $a \in \ker h_{n-1}(\partial C \hookrightarrow K)$ and $C \subset K$ are equivalent.*

Proof: We consider first the case where C is a PL-submanifold of $\mathbb{R}^n$ and then let $a = [\partial C]$ be the element generated in $h_{n-1}(\partial C)$ by ∂C. Clearly, $a \in \ker h_{n-1}(\partial C \hookrightarrow C)$. Moreover, if $K \supset \partial C$ is a compactum in $\mathbb{R}^n$ and $z_0 \in C \setminus K$, then we denote by C' the component of C containing z_0 and by $\partial_0 C'$ the outer boundary of C', and find a compact polyhedron $L \supset K$ which by a sequence of collapses retracts to a set L' of dimension $n-1$ containing $\partial_0 C'$. We request also that L contains $C \setminus C'$ and each of the bounded components of the complement of C. (Any sufficiently large ball with a ball in $C' \setminus K$ removed will do for L; see [R-S].) Then $[\partial_0 C'] \notin \ker h_{n-1}(\partial C \hookrightarrow L')$ by Lemma 1 and $a - [\partial_0 C'] \in \ker h_{n-1}(\partial C \hookrightarrow L)$. Consequently, $a \notin \ker h_{n-1}(\partial C \hookrightarrow L) \supset \ker h_{n-1}(\partial C \hookrightarrow K)$.

In the general case let $\{C_k : k \in N\}$ and $\{D_k : k \in N\}$ be sequences of PL-submanifolds such that $\bigcup\{C_k : k \in N\} = \operatorname{int} C$, $\bigcap\{D_k : k \in N\} = C$ and $C_k \subset \operatorname{int} C_{k+1} \subset D_{k+1} \subset \operatorname{int} D_k$ for each $k \in N$. For every $k \in N$ take an $a_k = [\partial C_k] \in h_{n-1}(\partial C_k)$ as above and write E_k for $D_k \setminus \operatorname{int} C_k$ and b_k for $h_{n-1}(\partial C_k \hookrightarrow E_k)(a_k)$. It is easy to see that $h_{n-1}(E_{k+1} \hookrightarrow E_k)(b_{k+1}) = b_k$. As $\partial C = \bigcap\{E_k : k \in N\}$ there is an $a \in h_{n-1}(\partial C)$ such that $h_{n-1}(\partial C \hookrightarrow E_k)(a) = b_k$ for every k. Then $h_{n-1}(E_k \hookrightarrow D_k)(b_k) = h_{n-1}(\partial C_k \hookrightarrow D_k)(a_k) = 0$ for every k, and hence $h_{n-1}(\partial C \hookrightarrow C)(a) = 0$. Conversely, if K is a compactum in $\mathbb{R}^n$ containing ∂C and $h_{n-1}(\partial C \hookrightarrow K)(a) = 0$ then, $\forall k \in N$,

$$\begin{aligned} 0 = h_{n-1}(\partial C \hookrightarrow K \cup E_k)(a) &= h_{n-1}(E_k \hookrightarrow K \cup E_k)(b_k) \\ &= h_{n-1}(\partial C_k \hookrightarrow K \cup E_k)(a_k). \end{aligned}$$

Consequently, $K \cup E_k \supset C_k \supset C_l$ whenever $k > l$. Since $\bigcap\{E_k : k > l\} = \partial C \subset K$ we get $K \supset C_l$ for every l. Hence $K \supset \bigcup C_l \cup \partial C = C$. ∎

Proof of Theorem 1: Let us consider the set

$$\widetilde{C} = \bigcup\{F(x) \times \{x\} : x \in C\} \subset Y \times \mathbb{R}^n.$$

We let $p\colon \widetilde{C} \to Y$ and $q\colon Y \times \mathbb{R}^n \to \mathbb{R}^n$ denote projections and write $\widetilde{\partial C}$ for $q^{-1}(\partial C) \cap \widetilde{C}$. Since F is n-acyclic $q|\widetilde{C}$ has n-acyclic point-inverses; thus by Vietoris' theorem $h_{n-1}(q|\widetilde{C})\colon h_{n-1}(\widetilde{C}) \to h_{n-1}(C)$ and $h_{n-1}(q|\widetilde{\partial C})\colon h_{n-1}(\widetilde{\partial C}) \to h_{n-1}(\partial C)$ are isomorphisms. Let $a \in h_{n-1}(\partial C)$ be as asserted in Lemma 2 and $\tilde{a} \in h_{n-1}(\widetilde{\partial C})$ correspond to a under the isomorphism $h_{n-1}(q|\widetilde{\partial C})$; then $\tilde{a} \in \ker h_{n-1}(\widetilde{\partial C} \hookrightarrow \widetilde{C})$. Letting b denote the image of $\tilde{a}$ under $h_{n-1}(p|\widetilde{\partial C}\colon \widetilde{\partial C} \to F(\partial C))$ we infer that $b \in \ker h_{n-1}(F(\partial C) \hookrightarrow F(C))$ and hence $b = 0$ by Lemma 1.

Now consider the compact subset Z of $Y \times C$,

$$Z = \bigcup\{\{y\} \times G(y)\}\colon y \in F(\partial C)\} \supset \widetilde{\partial C}.$$

The projection $Z \to F(\partial C)$ has n-acyclic point-inverses. Applying Vietoris' theorem again we infer that the image of $\tilde{a}$ in $h_{n-1}(Z)$ is mapped to b under the isomorhism $h_{n-1}(Z \to F(\partial C))$ and thus equals 0. Since $a = h_{n-1}(q)(\tilde{a})$ it follows that $a \in \ker h_{n-1}(\partial C \hookrightarrow q(Z))$. The properties of a ensure that $q(Z) \supset C$ and so there exists a point $y_0 \in F(\partial C)$ such that $x_0 \in G(y_0)$. ∎

In what follows we write $\operatorname{conv} P$ for the convex hull of a set $P \subset \mathbb{R}^n$.

Remark 2: Carathéodory's theorem [Ro] asserts that if $x \in \operatorname{conv} P$ then $x \in \operatorname{conv} P_0$ for some subset P_0 of P which is affinely independent and hence of cardinality $\leq n+1$; this implies that $\operatorname{conv} P$ is compact whenever P is. ∎

COROLLARY 1: *If $F\colon C \to Y$ is as above then there exists a set $C_0 \subset \partial C$ and a point $y_0 \in Y$ such that $C_0 \subset F^{-1}(y_0)$ and $x_0 \in \operatorname{conv} C_0$.*

Proof: Let $G(y) = \operatorname{conv}(F^{-1}(y) \cap \partial C)$ for $y \in F(\partial C)$. One shows in a straightforward manner that G is u.s.c. Hence, we may take $C_0 = F^{-1}(y_0) \cap \partial C$, where y_0 is given by Theorem 1. ∎

Remark 3: The above Corollary remains valid if the operation $A \mapsto \operatorname{conv} A$ is replaced by any other one $A \mapsto \operatorname{co} A$ which is n-acyclic, u.s.c. and defined on a sufficiently large family $\mathcal{F}$ of compacta; by this we mean that the following conditions are satisfied:

(i) For every $A \in \mathcal{F}$ the set $\operatorname{co} A \subset \mathbb{R}^n$ is compact and satisfies $h(\operatorname{co} A) = 0$ and $\operatorname{co} A \supset A$.

(ii) If $\{A_0, A_1, ...\} \subset \mathcal{F}$ and $\bigcup A_i \times \{1/i\} \cup A_0 \times \{0\}$ is compact, then $\bigcup \operatorname{co} A_i \times \{1/i\} \cup \operatorname{co} A_0 \times \{0\}$ is compact.

(iii) $A \in \mathcal{F}$ whenever both A is compact and $A \subset F^{-1}(y) \cap \partial C$ for some $y \in Y$. ∎

Remark: Theorem 1 remains valid if F is defined only on ∂C, rather than on C, and the condition $\dim(F(C) \setminus F(\partial C)) \leq n-1$ is replaced by the following one: $h_{n-1}(F)(a) = 0$ for some $a \in h_{n-1}(\partial C)$ as asserted by Lemma 2. (Keeping notation of the proof of Theorem 1, $h_{n-1}(F)$ is defined as $h_{n-1}(p|\widetilde{\partial C}) \circ (h_{n-1}(q|\widetilde{\partial C}))^{-1}$; then $h_{n-1}(F)(a) = b$.) The above condition is satisfied e.g. when $\dim(F(\partial C)) \leq n-2$. ∎

In Corollary 1, one may always take for C_0 a set of cardinality $\leq n+1$; see Remark 2. In general this inequality cannot be improved; to see this consider the natural simplicial map of the barycentric subdivision of an n-simplex Δ onto the join of the barycenter of Δ and of the $(n-2)$-skeleton of Δ. In the most intuitive special case below one can however decrease the cardinality of C_0 significantly, reducing convex hulls to segments:

PROPOSITION 1: *If $f\colon C \to Y$ is a continuous function and Y is an $(n-1)$-manifold, then there exist $x', x'' \in \partial C$ such that $f(x') = f(x'')$ and $x_0 \in \operatorname{conv}\{x', x''\}$.*

Proof: We consider first the case where C is a PL-manifold with boundary and $x_0 \in \operatorname{int} C$. Let B be a n-ball in $\mathbb{R}^n$ such that $x_0 \in \operatorname{int} B \subset B \subset \operatorname{int} C$. The radial (with respect to x_0) projection $\pi\colon \partial C \to \partial B$ induces a nonzero homomorphism of $(n-1)$-homology groups with coefficients $\mathbb{Z}_2$. $f|\partial C$ is extendable over C and so induces the trivial homomorphism of $(n-1)$-homology groups with coefficients $\mathbb{Z}_2$. By a generalization of the Borsuk–Ulam theorem due to J. Olędzki (see [Ol, Theorem 3.2]), there exist $x', x'' \in \partial C$ such that $f(x') = f(x'')$ and $\pi(x')$ and $\pi(x'')$ are antipodal points on ∂B. Then the segment joining x' and x'' contains x_0.

In the general case either $x_0 \in \partial C$, and then we take $x' = x'' = x_0$, or for every integer k we consider a compact n-manifold with boundary, C_k, such that $x_0 \in \operatorname{int} C_k$ and ∂C_k lies in the $\frac{1}{k}$-neighbourhood of ∂C in C. Let x'_k and x''_k satisfy the assertion with C replaced by C_k. Taking limits of converging subsequences gives the desired points x', x''. ∎

Remark: In the case $Y = \mathbb{R}^{n-1}$ the above statement was established in [Si] (for $n \leq 3$) and in [Jo] (arbitrary n). ∎

Question: In Corollary 1 assume additionally that F is single-valued and Y is an $(n-1)$-dimensional polyhedron. Under what conditions on Y can C_0 be taken to consist of at most of r points, where $r \leq n$ is given? Is the inequality $\max_{y \in Y} \operatorname{rank} \tilde{H}^{n-1}(Y, Y \setminus \{y\}; \mathbb{Z}) < r$ a sufficient condition? ∎

2. A separation theorem for families of convex functions and a system of inequalities

Here we use results of §1 to prove the result on separation of functions outlined in the introduction and derive from the latter Proposition 2 needed in §3. Separation theorems for families of functions were present, sometimes implicitly, in several papers on game theory (including ones on related topics; see [Bl] and [So]) and we hope our result may have some further applications to this theory and be also of independent interest. The reader aiming mainly at the application to section 3 can however pass directly to the formulation of Proposition 2 and then to the Appendix and skip the theorem.

In this section P and Q denote certain fixed compact convex sets in an Euclidean space, $[P]$ and $[Q]$ their affine spans and $\mathcal{L} = \mathcal{L}(Q)$ the $(\dim Q + 1)$-dimensional Banach space of all affine functionals on $[Q]$. We also fix a family $\{b_v\}_{v \in [P]}$ of real convex functions on $[Q]$ which is continuous (i.e. the induced function on $[P] \times [Q]$ is continuous). We say that a $\varphi \in \mathcal{L}$ **separates** (resp. **tightly separates**) a given member b_v of such a family from a function $a\colon Q \to \mathbb{R}$ if $a \leq \varphi|Q \leq b_v|Q$ and φ supports b_v at some point q of Q (resp. such that, moreover, $\psi(q) = b_v(q)$ whenever $\psi \in \mathcal{L}$ separates b_v from a); here, supporting means that the graph of φ is a supporting hyperplane for the epigraph of b_v at $(q, b_v(q))$.

The main result of this section is:

THEOREM 2: *Suppose additionally that P is a finite polyhedron of the same dimension as Q, and*

(1) $\forall T > 0\ \exists M > 0$ *such that* $\|v_1 - v_2\| < M$ *whenever* $\exists \varphi_1, \varphi_2 \in \mathcal{L}$ *supporting, resp.,* b_{v_1}, b_{v_2} *at points of Q and such that* $\varphi_1 - \varphi_2$ *is a constant in* $(-T, T)$.

Then, given $p_0 \in P$ *and* $a\colon Q \to \mathbb{R}$ *with* $b_v|Q \geq a$ $\forall v \in [P]$*, either* b_{p_0} *may be separated from* a *or there exist a set* $V \subset [P]$ *and a* $\varphi \in \mathcal{L}$ *such that, with* $r\colon [P] \to P$ *denoting the nearest-point retraction, we have*

(2) $p_0 \in \operatorname{conv} r(V)$ *and,* $\forall\, v \in V$, φ *tightly separates* b_v *from* a.

Remark: (a) The additional assumption that P be a polyhedron is irrelevant, provided $p_0 \in \operatorname{int} P$. (We skip the proof.)

(b) The sole reason for requesting that the functions b_v be defined on $[Q]$, rather than merely on Q, is to make supporting at points of ∂Q well defined. The following lemma and its proof show that for locally equi-lipschitz families $\{b_v\colon Q \to \mathbb{R}\}_{v\in[P]}$ the formulation of the theorem may be left intact if one interprets "φ supports b_v at $q \in Q$" in the preceeding definitions as $\varphi \in \operatorname{conv} \Psi(v,q)$, where Ψ is defined below: ∎

LEMMA 3: *Let* $\{b_v\}_{v\in[P]}$ *be a continuous family of real convex functions on* Q*. Then the following conditions are equivalent:*

(a) *There exists a continuous family* $\{\bar{b}_v\colon [Q] \to \mathbb{R}\}_{v\in[P]}$ *of convex extensions of the* b_v*'s;*

(b) *The family* $\{b_v\}_{v\in[P]}$ *is locally equi-lipschitzian;*

(c) *There exists a continuous family* $\{\widetilde{b}_v\colon [Q] \to \mathbb{R}\}_{v\in[P]}$ *of convex extensions of the* b_v*'s such that for any other such a family* $\{\bar{b}_v\}_{v\in[P]}$ *if a* $\varphi \in \mathcal{L}$ *supports a certain* $\widetilde{b}_v$ *at a point* $q \in Q$*, then it supports* $\bar{b}_v$ *at* q*.*

Proof: To show that (b) implies (c) consider the multifunction $\Psi\colon [P] \times Q \to \mathcal{L}$ defined by:

$$\Psi = \text{closure of } \{(v,q,\varphi) \in [P] \times \operatorname{int} Q \times \mathcal{L}\colon \varphi \text{ supports } b_v \text{ at } q\}$$

and for $(v,q) \in [P] \times [Q]$ let $\widetilde{b}_v(q) := \sup\{\varphi(q)\colon \varphi \in \Psi(v,q') \text{ for some } q' \in Q\}$. ∎

The proof of the theorem splits into the following steps:

I: *An additional claim and additional assumptions.* Applying a vertical shift we may assume that $a > 0$. Let $T > 0$ be such that

(3) $a(Q) \subset (0,T)$.

We will prove the result with the following assertion added:

(4) each vector in V has norm $\leq R$, where $R > 0$ is independent of p_0 and of a (as long as (3) holds).

It suffices to prove the result so altered in the special case where a is piecewise-linear. In fact, if (a_n) is any sequence of pl-functions satisfying

$$a - 1/n \leq a_n \leq a \quad \text{and} \quad a_n(Q) \subset (0,T), \quad \forall n,$$

then the sequence formed of solutions to the problem with a replaced by a_n has an accumulation point which does the job for the original data. Similarly, we may restrict our attention to the case where $p_0 \notin \partial P$.

Thus we assume we are in the special case described and set:

$$A = \operatorname{conv}\{(q,t) \in Q \times \mathbb{R}\colon t \leq a(q)\} \quad \text{and} \quad \widehat{a}(q) = \sup\{t\colon (q,t) \in A\}.$$

Then, A is a convex polytope and $\widehat{a}$ is the concave envelope of a. With $S = \{q \in Q\colon (q, \widehat{a}(q))$ is a vertex of $A\}$ we have

$$|S| < \infty \text{ and } \widehat{a}(s) = a(s) \leq b_v(s), \forall v \in [P], \quad \forall s \in S. \tag{5}$$

II: *Introducing a certain multifunction.* This part is influenced by S. Sorin's considerations of a family of functions of a single real variable; see pp. 201–203 of [So]. We define for $v \in [P]$:

$$z(v) = \max_{q \in Q}(\widehat{a}(q) - b_v(q)).$$

This is a continuous function and, by (3) and the inequality $b_v|Q \geq a \geq 0$,

$$z \leq T. \tag{6}$$

For $v \in [P]$ we write:

$$\Phi(v) = \{\varphi \in \mathcal{L}\colon \varphi \text{ separates } b_v + z(v) \text{ from } a\}.$$

Then $\Phi(v) \neq \emptyset$ by the separation theorem [Ro] and Φ takes values in non-empty convex compacta and is u.s.c. We'll show that

(7) If $\varphi \in \Phi(v)$ and $z(v) > 0$ then $|\{s \in S\colon \varphi(s) = a(s)\}| \geq 2$.

To see this fix a $q \in Q$ such that $\widehat{a}(q) = b_v(q) + z(v)$ and note that (5) and the assumption $z(v) > 0$ force $q \notin S$. Therefore, the smallest face of A containing $(q, \widehat{a}(q))$ has at least 2 vertices, and they all lie on the graph of φ.

It follows from (7) that $\Phi(\{v \in [P]: z(v) > 0\})$ is a subset of

$$H = \bigcup \{\{\varphi \in \mathcal{L}: \varphi(s_i) = a(s_i) \text{ for } i = 1, 2\}: s_1, s_2 \in S, s_1 \neq s_2\},$$

a finite union of codimension two hyperplanes in $\mathcal{L}$, and hence is of dimension $\leq \dim(Q) - 1$. (Although this is not needed later we note that the set considered is actually contained in the $(\dim Q - 1)$-skeleton of the canonical CW-subdivision of the space $\{\varphi \in \mathcal{L}: \varphi \geq \widehat{a} \text{ and } \varphi(s) = \widehat{a}(s) \text{ for some } s \in S\}$, and that the latter can be shown to be homeomorphic to a $\dim Q$-cell.)

III: *Completing the proof.* If $z(p_0) \leq 0$ then we are done by the standard separation theorem for convex sets [Ro]. We hence assume $z(p_0) > 0$ and define for every compact set $K \subset [P]$:

$$\operatorname{co} K = \{r(v)t + v(1 - t): v \in K \text{ and } t \in [0, 1]\} \cup \operatorname{conv} r(K).$$

Then $\operatorname{co} K$ is deformable to $\operatorname{conv} r(K)$ and thus is contractible. Applying Remarks 1 and 3 with

$$C = \text{closure of } \{v \in [P]: \|v\| < R \text{ and } z(v) > 0\},$$

where the value of R will be chosen later, we get a set $V \subset [P]$ and a $\varphi \in \mathcal{L}$ such that $p_0 \in \operatorname{conv} r(V)$ and:

(8) $\varphi \in \bigcap\{\Phi(v): v \in V\}$;

(9) for each $v \in V$, either $z(v) = 0$ or $\|v\| = R$ and $z(v) \geq 0$.

By (8) and our definitions (2) holds whenever no $v \in V$ satsfies $\|v\| = R$. Hence it remains to show that if R exceeds a certain number (not depending on a and on p_0) then $\|v\| < R$, $\forall v \in V$.

To this end suppose $v_0 \in V$ satisfies $\|v_0\| = R$ and write $w(v) := v - r(v)$ for $v \in [P]$ and $W(F) := \{w(v): v \in r^{-1}(F)\}$ for $F \subset P$. Let $\mathcal{F}$ be a collection of open faces of P such that $\bigcup \mathcal{F}$ is contained in no proper face of P. Then $\bigcap\{W(F): F \in \mathcal{F}\} = \{0\}$, so for every collection $\{w\} \cup \{w_F\}_{F \in \mathcal{F}}$ of non-zero vectors satisfying $w_F \in W(F) \forall F \in \mathcal{F}$ there exists an $F \in \mathcal{F}$ such that $\angle(w, w_F) \geq \alpha$, where α is an angle in $(0, \pi/2)$ that depends only on P. (This is because the unit sphere is compact and there are only finitely such families $\mathcal{F}$.) Now, since $p_0 \in \operatorname{int} P \cap \operatorname{conv} r(V)$, the set $r(V)$ is contained in no proper face of P and there exists a $v_1 \in V$ such that either $\angle(w(v_0), w(v_1)) \geq \alpha$ or $v_1 \in \operatorname{int} P$. With $D: = \sup_{p \in P} \|p\|$ we have $\|w(v_1) - w(v_0)\| \geq \|w(v_0)\| \sin(\alpha) \geq (R - D)\sin(\alpha)$ and

$$\|v_1 - v_0\| \geq (R - D)\sin(\alpha) - 2D.$$

Let $\varphi_i: = \varphi - z(v_i)$; then φ_i supports b_{v_i} and $|z(v_1) - z(v_0)| \leq T$, by (6) and the fact that $z(v_i) \geq 0$. Hence the desired conclusion follows from (1). ∎

We now prove the result on a system of inequalities needed in section 3. Below, I and K are finite sets and $\Delta := \Delta(K)$; then $[\Delta]$, the affine span of Δ in $\mathbb{R}^K$, equals $\{x \in \mathbb{R}^K : x \cdot e = 1\}$. In the subsequent proofs we sometimes canonically identify vectors in $\mathbb{R}^K$ with restrictions to $[\Delta]$ of the functionals they induce, and thus $\mathbb{R}^K$ with the set $\mathcal{L}$ of all affine functionals on $[\Delta]$ and scalar multiples of e with constant functions on $[\Delta]$.

PROPOSITION 2: *Let $a: \Delta(K) \to \mathbb{R}$ and $h: \Delta(I) \times \Delta(K) \to \mathbb{R}^K$ be continuous functions such that*

(10) *h is affine with respect to the variable $\sigma \in \Delta(I), \forall p \in \Delta(K)$, and*

(11) *$\forall p, q \in \Delta(K)\ \exists \sigma \in \Delta(I)$ such that $h(\sigma, p) \cdot q \geq a(q)$.*

Then, given $p_0 \in \Delta(K)$, there exist a set $P_0 \subset \Delta(K)$ of cardinality $\leq |K|$ and vectors $\sigma_p \in \Delta(I)$ $(p \in P_0)$ and $\varphi \in \mathbb{R}^K$ such that

(12) *$\varphi \cdot q \geq a(q),\ \forall q \in \Delta(K)$;*

(13) *$p_0 \in \operatorname{conv} P_0$;*

(14) *$\forall p \in P_0\ \forall k \in K$ we have $\varphi^k \geq h^k(\sigma_p, p)$, with equality occurring in place of $\geq$ whenever $p^k > 0$.*

Proof: For $p \in \Delta$ we let

$$b_p(q) := \max_{\sigma} h(\sigma, p) \cdot q = \max_{i \in I} h(e_i, p) \cdot q \quad \text{for } q \in [\Delta].$$

We claim that (cf. [So, p. 203]):

(15) If a $\varphi \in \mathcal{L}$ supports b_p then $\varphi = h(\sigma, p)$ for some $\sigma \in \Delta(I)$.

In fact, each maximal proper face of the epigraph B of b_p is contained in a hyperplane of $\mathbb{R}^K \times \mathbb{R}$ given for some $i \in I$ by $\{(q,t) \in [\Delta] \times \mathbb{R}\colon t = h(e_i,p)\cdot q\}$. Take an $x \in B \cap \{(q,\varphi(q))\colon q \in [Q]\}$ and let I_0 denote the set of all indices $i \in I$ corresponding to maximal proper faces of B that contain x. Then by a reformulation of Farkas' lemma [Ro] it follows that for some $\sigma \in \Delta(I_0) \subset \Delta(I)$ we have $\varphi = \sum_{i\in I_0} \sigma(i)h(e_i,p) = h(\sigma,p)$, as desired.

For $k \in K$ let $w_k = |K|^{-1}e - e_k$ be the outward vector normal to the maximal proper face of Δ opposite to e_k. The space $E\colon = [\Delta]-[\Delta] = \{x \in \mathbb{R}^K\colon x\cdot e = 0\}$ is a union of $|K|$-many cones, each generated by all vectors w_k but one. Hence each $v \in E$ can be uniquely written in the form $\Sigma\lambda_k w_k$, where all the λ_k's are ≥ 0 and one of them is 0, and we define $Lv = \Sigma\lambda_k e_k$. On any of our cones L is an isomorphism onto a "quadrant" $\{x \in \mathbb{R}^K\colon x \geq 0 \text{ and } x^k = 0\}$. It follows that $\lim_{\|v'-v''\|\to\infty} \|Lv' - Lv''\| = \infty$, whence the family $\{b_v\}_{v\in[\Delta]}$ defined by $b_v := b_{r(v)} + L(v - r(v))$ satisfies condition (1). Applying Theorem 2 and (15) we get a set $V \subset [\Delta]$ and a $\varphi \in \mathbb{R}^K$ satisfying (12) and such that $p_0 \in \operatorname{conv} r(V)$ and

$$(14)' \qquad \forall v \in V\ \exists\sigma_v \in \Delta(I) \text{ with } \varphi = h(\sigma_v, r(v)) + L(v - r(v)).$$

We write $P_0 = r(V)$ and for every $p \in P_0$ choose a $v_p \in r^{-1}(p) \cap V$ and write $\sigma_p = \sigma_{v_p}$. It remains to show that if $v \in V$ and $k \in K$ satisfy $(r(v))^k > 0$ then $(L(v - r(v)))^k = 0$. However, we then have $v - r(v) = \Sigma\lambda_{k'}w_{k'}$ where all the $\lambda_{k'}$'s are ≥ 0 and $\lambda_k = 0$, so using the definition of L completes the proof. ∎

Appendix. A more direct proof of Proposition 2 can be given as follows. We write for $v \in [\Delta]$:

$$r(v)(k) := \max(v^k,0)/\sum\{v^l\colon\ v^l \geq 0\} \text{ and } \ u(v)(k) := |\min(v^k,0)|;$$

$$b_v(q) := \max_\sigma h(\sigma, r(v))\cdot q + u(v)\cdot q \qquad \text{for } q \in [\Delta].$$

Suppose we showed the existence of a set $V \subset \mathbb{R}^K$ and of a $\varphi \in \mathcal{L}$ such that $p_0 \in \operatorname{conv} r(V)$, $a \leq \varphi|\Delta$ and φ supports b_v, $\forall v \in V$. By (15) we then have

$$(14)'' \qquad \forall v \in V\ \exists\sigma_v \in \Delta \text{ with } \varphi = h(\sigma_v, r(v)) + u(v).$$

Thus letting $P_0 = r(V)$ and $\sigma_p = \sigma_{v_p}$, where $v_p \in r^{-1}(p) \cap V$ for $p \in P_0$, we easily see that (12)–(14) hold (note that $u(v)(k) = 0$ if $r(v)(k) > 0$).

To get a pair (V, φ) as above follow the proof of Theorem 2 taking $P = Q = \Delta$. Only the last paragraph of that proof needs to be simplified and modified as follows:

We show that it suffices to take $R = \max\{\|v\|: v \in [\Delta] \text{ and } \inf_k v(k) \geq -2T_1\}$, where $T_1 := T + \max_{\sigma,p} \|h(\sigma, p)\|$. (We have $R < \infty$ because $v \cdot e = 1$ for $v \in [\Delta]$.) In fact, suppose $\|v_0\| > R$ for some $v_0 \in V$ and let $l \in K$ be such that $v_0(l) < -2T_1$; then $u(v_0)(l) > 2T_1$. By (8) and (15) we infer that

$$(14)''' \qquad \forall v \in V \; \exists \, \sigma_v \in \Delta \text{ with } \varphi = h(\sigma_v, r(v)) + u(v) + z(v)e.$$

Hence $\varphi(l) > T_1$, by (9). However, as $p_0 \in \operatorname{conv} r(V) \cap \operatorname{int} \Delta$ there also exists a $v \in V$ such that $r(v)(l) > 0$; then $u(v)(l) = 0$ and $\varphi(l) \leq T_1$ by (14)''' and (6), a contradiction. ∎

Remark: Alternatively, one can introduce in a standard fashion variables $u^k \geq 0$ that allow to control (14):

$$\widetilde{\Delta} = \{(p, u) \in \Delta \times \mathbb{R}^K: \; u^k \geq 0 \text{ and } p^k u^k = 0 \text{ for all } k \in K\}.$$

It turns out that $\widetilde{\Delta}$ is homeomorphic to $\mathbb{R}^l$ for $l = |K| - 1$. (In fact, a homeomorphism $\widetilde{\Delta} \to \{x \in \mathbb{R}^K: x \cdot e = 1\}$ may be given by $(p, u) \mapsto (1 + \|u\|_1)p - u$ and its inverse by $v \mapsto (r(v), u(v))$; we skip the verification.) This allows one to apply Theorem 1 and to proceed as above, with $\widetilde{\Delta}$ playing the role of $[\Delta]$ and the projections $\widetilde{\Delta} \to \Delta$ and $\widetilde{\Delta} \to \mathbb{R}^K$ playing the role of the functions r and u, respectively. ∎

3. Undiscounted repeated two-person games of incomplete information on one side

One-shot games of incomplete information on one side were first introduced by J. Harsanyi [H] and the infinitely repeated ones of this type by R. Aumann and M. Maschler [Au-Ma1]; further basic results relevant to this section were obtained in [Au-Ma-St] and [So]. A brief description is as follows. A game between two players named $\mathcal{A}$ and $\mathcal{B}$ proceeds in infinitely many successive stages. In the 0-th stage a k is chosen from a finite set K of "states of nature" according to a probability distribution $p_0 \in \operatorname{int} \Delta(K)$. In any subsequent stage each of the players selects a "pure action" from a finite set I (for $\mathcal{A}$) or J (for $\mathcal{B}$), gaining a stage-payoff $A_k(i,j)$ (for $\mathcal{A}$) or $B_k(i,j)$ (for $\mathcal{B}$) which depends only on the pure actions $i \in I$ and $j \in J$ selected in this stage and on the "true state of nature" k, chosen at stage 0. (The game is undiscounted because there is no geometrically decreasing weighting of the consecutive stage-payoffs.) The families $\{A_l\}_{l\in K}$ and $\{B_l\}_{l\in K}$ and the distribution p_0, as well as the rules of the game, are given to the players before the game starts as their initial common knowledge. At any stage the players also know the pure actions both of them took on preceding stages and $\mathcal{A}$ (but not $\mathcal{B}$) knows the outcome $k \in K$ of the 0-stage.

In this section we prove that the games above admit an equilibrium, and in fact one of a very special type studied in earlier work of other authors. We describe this equilibrium later but at this moment we would like to say that its nature is such that for the players to have a chance of making use of it, it is natural to assume that pre-play communication is admitted by the rules of the game. However, even if inaccessible, this equilibrium remains to be one if no communication is permitted, and we leave aside the question whether and how can the players actually reach an equilibrium with pre-play communication prohibited.

For a mathematical setup, let $I^* = \Delta(I)$ and $J^* = \Delta(J)$ be the spaces of mixed strategies of the respective players and let Δ denote $\Delta(K)$. A **behaviour strategy** of a player is a sequence of functions: $(K \times (I \times J)^n \to I^*)_{n\geq 0}$ for $\mathcal{A}$ and $((I \times J)^n \to J^*)_{n\geq 0}$ for $\mathcal{B}$. (See [Bl], [Ku], [Au-Ma-St].) Thus for each stage a behaviour strategy just gives a recipe determining a probability measure on the set of player's pure actions based on the appropriate part of the past history of the game; it shouldn't be confused with elements of the sets I or J or I^* or J^* (also called "strategies" or "mixed strategies"). A **mixed behaviour strategy** is* a finite list of behaviour strategies of a player and a probability measure on the list's indexing set; for player $\mathcal{A}$, this measure may depend on the true state of nature. (After stage 0, the player under consideration performs a lottery according to this probability distribution and decides to apply throughout the resulting behaviour strategy. By a general theorem of H. W. Kuhn ([Ku], [Se]) or by a direct argument the use of such strategies can be eliminated in favour of behaviour strategies, but doing so would complicate our exposition.)

Let $H = (I \times J)^\infty$. A pair of behaviour strategies, s for $\mathcal{A}$ and t for $\mathcal{B}$,

* Our terminology and definitions differ slightly from ones used in other sources (cf. [Se]), but lead to equivalent notions.

determines on $K \times H$ a probability measure $\mu_{s,t}$ such that the probability of the K-coordinate to be l is $p_0(l)$ and, conditionally on the previous coordinates, the measure on the n-th factor of $(I \times J)^\infty$ is the product of the measures determined at the n-th stage by the strategies in question, given the previous information. This definition extends to the case where s and t are mixed behaviour strategies to yield a measure $\mu_{s,t}$ defined on $K \times L_s \times L_t \times H$, where L_s and L_t are the indexing sets for the lists of strategies for s and t, respectively. Given such a pair (s,t) one may consider the sequence of arithmetic means of the consecutive stage-payoffs of a player $\mathcal{C} = \mathcal{A}$, $\mathcal{B}$, integrated with respect to the induced conditional measure on a given Borel subset S of $K \times L_s \times L_t \times H$; its limit superior (resp. its limit, if existing) will be called $\mathcal{C}$'s valuation of (s,t) on S and denoted $V_{\mathcal{C}}^S(s,t)$ (resp. $E_{\mathcal{C}}^S(s,t)$). For z in the disjoint union of the sets K, L_s and L_t we agree $V_{\mathcal{C}}^z$ to mean $V_{\mathcal{C}}^S$, where S is given by the requirement that the corresponding coordinate be z; also, we drop the superscript S if $S = K \times L_s \times L_t \times H$ and identify $K \times L_s \times L_t \times H$ with $K \times L_s \times H$ (resp. with $K \times H$) if L_t is a singleton (resp. if L_s is, too). We have $V_{\mathcal{C}}^{\bigcup S_n} \leq \max_n V_{\mathcal{C}}^{S_n}$.

For $p \in \Delta$ we write $A(p) = \sum_{k \in K} p^k A_k$ and $B(p) = \sum_{k \in K} p^k B_k$ and, using the min-max theorem, we define $a(p)$ and $b(p)$ by the formulas:

$$a(p) := \max_{\sigma \in I^*} \min_{\tau \in J^*} \sigma A(p) \tau \quad \text{and} \quad b(p) := \min_{\sigma \in I^*} \max_{\tau \in J^*} \sigma B(p) \tau = \max_{\tau \in J^*} \min_{\sigma \in I^*} \sigma B(p) \tau.$$

We now proceed to describe S. Sorin's notion of an "independent and 2 safe joint plan equilibrium", which is a specification of a weakened form of the "joint plan equilibrium" of R. Aumann, M. Maschler and R. Stearns. The discussion below and in the appendix to this section is motivated by that in [Au-Ma-St] and in [So] but differs in being independent from the 0-sum case studied in [Au-Ma1] (a paper nearly inaccessible in its original edition but to be reprinted in [Au-Ma2]) and in taking closer to the surface the role of condition (E1) to follow, crucial for both the need and the possibility of using results from §2. In general, by an **equilibrium** one understands a pair (s^*, t^*) of mixed behaviour strategies such that $E_{\mathcal{A}}(s^*, t^*)$ and $E_{\mathcal{B}}(s^*, t^*)$ exist and satisfy $E_{\mathcal{A}}(s^*, t^*) \geq V_{\mathcal{A}}(s, t^*)$ and $E_{\mathcal{B}}(s^*, t^*) \geq V_{\mathcal{B}}(s^*, t)$ for all behaviour strategies s, t of respective players. (A result of S. Hart [Ha] gives a complete characterization of all equilibrium valuation pairs for this game; see also [Au-Ha].) An **independent and 2 safe joint plan** is a very specific kind of an equilibrium which may be described as follows.

$\mathcal{B}$ starts by finding, if it can, a vector $\varphi \in \mathbb{R}^K$, a finite subset P_0 of Δ and for every $p \in P_0$ elements $\sigma_p \in I^*$, $\tau_p \in J^*$ and $\lambda_p \in (0,1]$ such that

(E1) $p_0 = \sum_{p \in P_0} \lambda_p p$ and $\sum_{p \in P_0} \lambda_p = 1$;

(E2) $\varphi \cdot q \geq a(q) \quad \forall\, q \in \Delta$;

(E3) $\sigma B(p) \tau_p \geq b(p), \quad \forall\, p \in P_0 \quad \forall\, \sigma \in I^*$;

(E4) $\varphi^k \geq \sigma_p A_k \tau_p \quad \forall\, p \in P_0 \quad \forall\, k \in K$, with $=$ occurring instead of $\geq$ whenever $p^k > 0$.

$\mathcal{B}$ also finds a bijection f of P_0 into a finite power of I (say, into I^l; one may assume $|I| \geq 2$) and for each $p \in P_0$ takes a function $h_p = (h_p^I, h_p^J)\colon \mathbb{N} \to I \times J$ such that

$$\lim_{n \to \infty} |\{m\colon m \leq n \text{ and } h_p(m) = (i,j)\}|/n = \sigma_p(i)\tau_p(j),\ \forall (i,j) \in I \times J.$$

$\mathcal{B}$ chooses to convey all this structure to $\mathcal{A}$ along with the following description of a pair (s^*, t^*).

$\mathcal{A}$'s mixed behaviour strategy s^* is a combination of certain behaviour strategies $s_p\,(p \in P_0)$ taken with weights $x_k^p = \lambda_p p^k / p_0^k$, where k is the true state of nature. (I.e., $\mathcal{A}$ chooses after stage 0 a $p \in P_0$ following this distribution x_k on P_0 and decides to use s_p throughout.) Here, s_p demands that $\mathcal{A}$'s move at a given stage m be $f(p)(m)$ if $m \leq l$ and be $h_p^I(m)$ if $m > l$ and $\mathcal{B}$ followed h_p^J in all preceeding stages $> l$, and that it be taken in accordance with a fixed probability distribution $\sigma_p' \in I^*$ for which $\max_{\tau \in J^*} \sigma_p' B(p) \tau = b(p)$ if neither of the above "if" conditions is satisfied. $\mathcal{B}$'s behaviour strategy t^* is to take arbitrary moves at the first l stages and at any next to play according to h_p^J if there exists a (unique) $p \in P_0$ such that the sequence of the first l moves of $\mathcal{A}$ gave a point $f(p)$ and $\mathcal{A}$ followed h_p^I in all stages $> l$ played up to then, or else to play according to a strategy t_φ given by the following consequence of a general result of D. Blackwell ([Bl, p.6], cf. [Au-Ma1] and [Au-Ma-St] for this particular application):

LEMMA 4: *Given $\varphi \in \mathbb{R}^K$ satisfying* (E2) *player $\mathcal{B}$ has a behaviour strategy t_φ such that for every behaviour strategy s of $\mathcal{A}$, every $k \in K$ and every finite sequence $x \in (I \times J)^n$ $(n \in \mathbb{N})$ one has $V_{\mathcal{A}}^{[k,x]}(s, t_\varphi) \leq \varphi^k$, where $[k,x]$ denotes $\{k\} \times \{(i_l, j_l)_{l \in \mathbb{N}} \in H\colon (i_1, j_1, \ldots, i_n, j_n) = x\}$.*

It can be shown that the pair (s^*, t^*), if it exists, is an equilibrium; see the appendix to this section and the references given there for a discussion making the role of conditions (E1) to (E4) easier to grasp. (These conditions correspond to what one gets by combining (i)–(iii) from [So, p. 197] with the requirement of [So, p. 199].) To establish the existence we need

LEMMA 5: *For every $\varepsilon > 0$ there exists a continuous map $g\colon D \to J^*$ such that $\sigma B(p)g(p) \geq b(p) - \varepsilon$ for all $(\sigma, p) \in I^* \times \Delta$.*

Proof: Write $v(\tau, p) = \min_\sigma \sigma B(p)\tau$ and $T(p) := \{\tau \in J^*\colon v(\tau,p) = b(p)\}$; then $v\colon J^* \times \Delta \to \mathbb{R}$ is continuous and concave in J^* for fixed p. Since b also is continuous, there exists a δ so that $\|p' - p\| \leq \delta$ implies $\|b(p') - b(p)\| \leq \varepsilon/2$ and $|v(\tau,p') - v(\tau,p)| \leq \varepsilon/2$ for every $\tau \in J^*$. Create a simplicial subdivision Δ_ε of Δ with mesh diameter less than δ. Create a piece-wise linear mapping $g\colon \Delta \to J^*$ so that if p is a vertex of Δ_ε then $g(p)$ is any member of $T(p)$; it follows easily that g is as desired. (This Lemma is a special case of a much more general topological statement, see [Hav].) ∎

Now we are able to prove:

THEOREM 3: *Every undiscounted infinitely repeated two-person game of incomplete information on one side has an independent and 2 safe joint plan equilibrium.*

Proof: We need to find a system satisfying (E2)–(E4) and such that $p_0 \in \operatorname{conv} P_0$. Given $\varepsilon > 0$ let g be as asserted in Lemma 5. Applying Proposition 2 with $h^k(\sigma,p) = \sigma A_k g(p)$ for $k \in K$ we get a set P_0 with $|P_0| \leq |K|$ and vectors: $\varphi \in \mathbb{R}^K$, $\sigma_p \in I^*$ and $\tau_p := g(p) \in J^*$ ($p \in P_0$) satisfying all the properties needed, except that $b(p)$ is replaced by $b(p) - \varepsilon$ in (E3). Taking a cluster point of these approximate solutions as $\varepsilon \to 0$ will give the system we seek, but its existence needs commenting for the case of the φ's (the remaining elements of the systems belong to certain apriori given simplices and thus have converging subsequences). However, as $p_0 \in \operatorname{int}\Delta \cap \operatorname{conv} P_0$, for every $k \in K$ there exists a $p \in P_0$ with $p^k \neq 0$. Hence condition (E4) yields a bound on $\|\varphi\|_\infty$ depending only on the payoff matrices, thus locating the φ's in a desired compact set. ∎

Appendix: The equilibrium property of independent and 2 safe plans

To make the role of condition (E1) more transparent we assume in greater generality that (E1) and the equality $x_k(p) = \lambda_p p^k/p_0^k$ are dropped and that an arbitrary system of vectors $x_k \in \Delta(P_0)$, $k \in K$, is used to define the pair of strategies as described before. We keep assuming (E2)–(E4), with "whenever $p^k > 0$" in (E4) replaced by "whenever $x_k(p) > 0$".

We first note that for all $(k,p) \in K \times P_0$ and any set $S \subset \{(k,p)\} \times H$ the deterministic strategies h_p^I and h_p^J satisfy:

(i) $E_{\mathcal{C}}^S(h_p^I, h_p^J) = \sigma_p C_k \tau_p$, where $C = A$ if $\mathcal{C} = \mathcal{A}$ and $C = B$ if $\mathcal{C} = \mathcal{B}$.

Next, we fix a $k \in K$ and a behaviour strategy s of $\mathcal{A}$ and define

$$S_k := \{k\} \times \{h \in H\colon h \text{ is such that } \mathcal{B} \text{ was given a cause to apply } t_\varphi\};$$

then

(ii) $V_{\mathcal{A}}^{S_k}(s, t^*) \leq \varphi^k$.

(We postpone the proof.) Modulo a μ_{s,t^*}-measure 0 set, $U_k := \{k\} \times H \setminus S_k$ is a disjoint union of the sets $U_k(p) := \{u \in \{k\} \times (I \times J)^\infty$: the first l of the I-coordinates of u give $f(p)$ and the $I \times J$-coordinates of u from stage $l+1$ on are described by $h_p\}$. Letting $y_k(p) = \mu_{s,t^*}(U_k(p))/\mu_{s,t^*}(U_k)$ we get by (i) and (E4):

$$E_{\mathcal{A}}^{U_k}(s, t^*) = \Sigma_p y_k(p) \sigma_p A_k \tau_p \leq \Sigma_p y_k(p) \varphi_k = \varphi_k.$$

If $s = s^*$ then above we have $=$ in place of $\leq$ and $\mu_{s,t^*}(S_k) = 0$. Hence

$$V_{\mathcal{A}}^k(s, t^*) \leq \max(V_{\mathcal{A}}^{S_k}(s, t^*), E_{\mathcal{A}}^{U_k}(s, t^*)) \leq \varphi_k = E_{\mathcal{A}}^k(s^*, t^*)$$

and $V_{\mathcal{A}}(s, t^*) \leq E_{\mathcal{A}}(s^*, t^*)$ by the arbitrareness of k.

To get information on $V_{\mathcal{B}}(s^*, .)$ write ν_p for $\Sigma_k p_0(k) x_k(p)$, the total probability of $\mathcal{A}$'s lottery to yield an outcome $p \in P_0$. Given such an outcome, the conditional probability of the true state of nature to be k is $\tilde{p}(k) := p_0(k) x_k(p)/\nu_p$; here $\nu_p > 0$ since $p_0 \in \operatorname{int} \Delta(K)$. We fix $p \in P_0$ and a behaviour strategy t of $\mathcal{B}$ and denote by μ the relative measure induced by $\mu_{s^*,t}$ on $K \times \{p\} \times H$. We also write $T = \bigcup T_n$, where $T_n := K \times \{p\} \times \{h \in H_p\colon h(m) \neq h_p(m)$ for some $m \in \mathbb{N}$ with $l < m \leq n\}$. The remaining discussion rests on the fact that, for each $k \in K$:

(iii) On T the probability of the true state of nature to be k is $\tilde{p}(k)$,

(iv) $V_{\mathcal{B}}^T(s^*, t) \leq \max_{\tau \in J^*} \sigma'_p B(\tilde{p}) \tau$.

Assuming (iii) and (iv) we note that (iii) keeps being valid with T replaced by $S := K \times \{p\} \times H \setminus T$, whence $E_{\mathcal{B}}^{S}(s^*,t) = \Sigma_k \tilde{p}(k)\sigma_p B_k \tau_p = \sigma_p B(\tilde{p})\tau_p$ by (i). When $t = t^*$ we have $\mu(T) = 0$, so by (E3) we infer that if $\tilde{p} = p$ then

$$V_{\mathcal{B}}^{p}(s^*,t) \leq \max(E_{\mathcal{B}}^{S}(s^*,t),\ V_{\mathcal{B}}^{T}(s^*,t)) \leq \sigma_p B(p)\tau_p = E_{\mathcal{B}}^{p}(s^*,t^*).$$

Thus (s^*,t^*) is an equilibrium whenever $\tilde{p} = p\ \forall p \in P_0$. Except for the justification of (ii)–(iv), the discussion may be concluded by invoking the easily verifiable lemma (for the "only if" part set $\lambda(p) := \nu_p$ for all $p \in P_0$):

LEMMA 6: *The condition $\tilde{p} = p\ \forall p \in P_0$ holds if and only if there exists a $\lambda \in \Delta(P_0)$ such that $p_0 = \Sigma_p \lambda(p)p$ and the vectors $x_k \in \Delta(P_0)$ satisfy $x_k(p) = \lambda(p)p(k)/p_0(k)$, $\forall k \in K, \forall p \in P_0$.*

To demonstrate (iii) and (iv) we fix $n \in \mathbb{N}$, split T_n as $K \times \{p\} \times X \times Y$, where X is the image of T_n under the projection to the product of the first n factors of $(I \times J)^\infty$, and from the definition of μ read off the conditional measures $\mu_Y^{k,x}$ on Y, μ_X^k on $(I \times J)^n$ and μ_K on K such that $\mu = \int_K \int_X \mu_Y^{k,x} d\mu_X^k(x) d\mu_K(k)$. Then, μ_K is given by $\tilde{p}$ and $\int_K B_k d\mu_K = B(\tilde{p})$. With (i_m, j_m) denoting the projection of $K \times P \times H$ onto the m-th factor of $H = (I \times J)^\infty$, the mean value on T_n of $\mathcal{B}$'s m-th stage payoff satisfies for $m > n$ (we treat I, J as subsets of $\Delta(I), \Delta(J)$):

$$r_{n,m} := (\mu(T_n))^{-1} \int_{T_n} i_m B_k j_m d\mu = (\mu(T_n))^{-1} \int_K \int_X \int_Y i_m B_k j_m d\mu_Y d\mu_X d\mu_K.$$

The strategies t and $s^*|K \times \{p\} \times H = s_p$ are independent of k; thus so are the measures $\mu_k := \int_X \mu_Y^{k,x} d\mu_X^k(x)$. Also, with respect to any of the measures $\mu_Y^{k,x}$ the random variables i_m and j_m are independent and i_m has distribution σ_p'. Thus $a := \int_{X \times Y} 1 d\mu_k$ and $\tau := \int_{X \times Y} j_m d\mu_k / a \in J^*$ are independent of k and integrations above yield $\int_X \int_Y$ equal to $a\sigma_p' B_k \tau$ and $r_{n,m} = \mu(T_n)^{-1} a \sigma_p' B(\tilde{p})\tau$. Similarly, for the characteristic function g_k of $\{(k,p)\} \times X \times Y$ we get $\int_{T_n} g_k d\mu = \tilde{p}(k)a$ and (hence) $\mu(T_n) = \int_{T_n} 1 d\mu = a$. Combining these we get (iii) and (iv) (first with T_n in place of T, where however n is arbitrary).

Inequality (ii) follows similarly, by using a filtration of S_k analogous to the filtration of T by the T_n's and then estimating

$$\int_Y i_m A_k j_m d\mu_Y^{k,x}(y) = V_{\mathcal{A}}^{[(k,x)]}(s, t_\varphi) \leq \varphi_k$$

in the partial integration. ∎

Remark: The result of Blackwell [Bl] admits for establishing additional properties of the strategy t_φ of Lemma 4 and thus also of the pair (s^*, t^*).

The interested reader should consult also [Au-Ma-St] and [So]. ∎

References

[Au-Ha] R. Aumann and S. Hart, *Bi-convexity and bi-martingales*, Israel Journal of Mathematics **54** (1986), 159–180.

[Au-Ma1] R. Aumann and M. Maschler, *Game Theoretic Aspects of Gradual Disarmament*, Mathematica V.1–V.55, Princeton N.J. (1966) (to be reprinted in [Au-Ma2]).

[Au-Ma2] R. Aumann and M. Maschler, *Repeated Games with Incomplete Information*, prepared for M.I.T. Press.

[Au-Ma-St] R. Aumann, M. Maschler and R. Stearns, *Repeated games of incomplete information: An approach to the non-zero-sum case*, Mathematica ST-143, Ch IV, 117–216, Princeton, N.J. (1968) (to be reprinted in [Au-Ma2]).

[Be] E. Begle, *The Vietoris mapping theorem for bicompact spaces*, Annals of Mathematics **51** (1950), 534–543.

[Bl] D. Blackwell, *An analog of the minimax theorem for vector payoffs*, Pacific Journal of Mathematics **6** (1956), 1–8.

[Bo] K. Borsuk, *Drei Sätze über die n-dimensionale Euklidische Sphäre*, Fundamenta Mathematicae **20** (1933), 177–190.

[ES] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press, 1952.

[Fo] F. Forges, *Repeated games of incomplete information: non-zero-sum*, in *Handbook of Game Theory*, Vol. 1 (R. J. Aumann and S. Hart, eds.), Elsevier Science Publishers, Amsterdam, 1992.

[Hars] J. Harsanyi, *Games with incomplete information played by Bayesian players, parts I–III*, Management Science **14** (1967/68), 159–182, 320–334, 486–502.

[Hart] S. Hart, *Non-zero-sum two-person repeated games with incomplete information*, Mathematics of Operations Research **10** (1985), 117–153.

[Hav] W. E. Haver, *A near-selection theorem*, General Topology and its Applications **9** (1978), 117–124.

[Jo] K. D. Joshi, *A non-symmetric generalization of the Borsuk–Ulam theorem*, Fundamenta Mathematicae **80** (1973), 13–33.

[Ku] H. W. Kuhn, *Extensive Games and the Problem of Information*, in *Contribution to the Theory of Games II*, Annals of Mathematics Studies **28** (1953), 193–216.

[Ol] J. Olędzki, *On a generalization of the Borsuk-Ulam theorem*, Bulletin de l'Académie Polonaise des Sciences **26** (1978), 157–162.

[Ro] R. T. Rockafeller, *Convex Analysis*, Princeton Univ. Press, 1970.

[R-S] C. P. Rourke and B.J. Sanderson, *Introduction to Piecewise-Linear Topology*, Springer-Verlag, Berlin, 1972.

[Se] R. Selten, *Reexamination of the perfectness concept for equilibrium points in extensive games*, International Journal of Game Theory **4** (1974), 25–55.

[Si] K. Sieklucki, *A generalization of the Borsuk–Ulam theorem on antipodal points*, Bulletin de l'Academie Polonaise des Sciences **17** (1969), 629–631.

[So] S. Sorin, *Some results on the existence of Nash equilibrium for non-zero-sum games with incomplete information*, International Journal of Game Theory **12** (1983), 193–205.

[Sp] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.